Name of College – S.S. College, Jehanabad.

Deptt – Mathematics

TOPIC – Adjoint, inverse of a Matrix A and Rank of Matrix (Cont-d)

Class – B.Sc I (HONS + Sub)

Time – 1.00 P.M to 1.45

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Show that if A be an $n\times n$ non-singular Matrix, then

$$(A^T)^{-1} = (A^{-1})^T$$

Proof :→ Let A be a non-singular Matrix of order n. thus, we have

$$|A| \neq 0$$

Since from property of determinant, we have $|A| = |A^T|$

We have $AA^{-1} = A^{-1}A = I$

On taking Transpose of the above, we get

$$(AA^{-1})^T = (A^{-1}A)^T = I^T$$

$$\Rightarrow (A^{-1})^T \cdot (A)^T = A^T \cdot (A^{-1})^T = I$$

$\because I^T = I$ and by reversal Law of Transpose

From the above, result we can say that -

$(A^{-1})^T$ is the inverse of A^T

i.e. $(A^T)^{-1} = (A^{-1})^T$

This result states that - "The inverse of the Transpose of a Matrix is the Transpose of the inverse of the matrix.

Show that - the inverse of an orthogonal Matrix is orthogonal or

if P is orthogonal so is P^{-1}

Proof :→ Let A be a square matrix which is orthogonal.

then $A^T A = I$ —(i) [$AA^T = A^TA = I \Rightarrow A$ is orthogonal matrix]

Let B be inverse of A

then $AB = BA = I$ —(ii) [$AA^{-1} = A^{-1}A = I$]

Now we are to show B is orthogonal

we have $(AB)^T(AB) = (B^TA^T)(AB)$

$= B^T(A^TA)B.$

$= B^T \, IB$

$= B^T(IB)$

$= B^TB$

Again, $(AB)^T(AB)$ using (ii)

$= I^T \, I$ $\quad I^T = I$

$= I \cdot I$

From Equation (iii) and (iv) we have

$B^T B = I \Rightarrow B$ is orthogonal

$\Rightarrow$ inverse of A is orthogonal

$\Rightarrow A^T$ is orthogonal

* When is in general 2×2 Matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible?

What then is its inverse —

Proof :→ We seek scalars x, y, z, t such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \begin{bmatrix} x & y \\ z & t \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

On applying Matrix Multiplication, we get -

$$\begin{bmatrix} ax + bz & ay + bt \\ cx + dz & cy + dt \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By the definition of Equality of two Matrices

$ax + bz = 1$ — (I) and $ay + bt = 0$ — (III)

$cx + dz = 0$ — (II) $cy + dt = 1$ — (IV)

both of which have co-efficient matrix A.

det $|A| = ad - bc$

Suppose $|A| \neq 0 \Rightarrow ad - bc \neq 0$

(I) × d − (II) × b

$$\Rightarrow adx - bcx = d \Rightarrow x = \frac{d}{ad-bc} = \frac{d}{|A|}$$

$$z = \frac{\begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-c}{|A|}$$

from (III) and (IV)

$$y = \frac{\begin{vmatrix} 0 & b \\ 1 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{0-b}{ad-bc} = -\frac{b}{ad-bc} = -\frac{b}{|A|}$$

$$t = \frac{\begin{vmatrix} a & 0 \\ c & 1 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{a}{ad-bc} = \frac{a}{|A|}$$

and the 2nd system of Equation has unique Solution

$$y = \frac{-b}{|A|} \quad t = \frac{a}{|A|}$$

Accordingly $A^{-1} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$

$$= \begin{bmatrix} \frac{d}{|A|} & -\frac{b}{|A|} \\ -\frac{c}{|A|} & \frac{a}{|A|} \end{bmatrix}$$

$$= \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In words When $|A| \neq 0$
the inverse of a 2×2 Matrix A can be obtained

(I) by interchanging the elements of the Main diagonal

(II) Taking the -ves of the other elements of secondary diagonal and Multiplying the Matrix by $\frac{1}{|A|}$

Rank of a Matrix

A $m \times n$ Matrix A is said to have rank r if it has at least one sub-matrix of order which is Non singular but all sub-matrices of order greater than r are singular.

usually Rank of Matrix A is denoted by $\rho(A)$ or $r(A)$ or Rank of A.

Nullity of a Matrix

If a Square Matrix of order n then $n - \rho(A)$ is called Nullity of the Matrix

= order of Matrix − rank of Matrix
